Unanswered Mathematical Questions Raised by Fractal Geometry

Benoît B. Mandelbrot
Yale University
New Haven, USA

Abstract

This text sketches diverse questions of pure mathematics that fractal geometry raised over the years. Some are broadly-based challenges. Others are fully-fledged conjectures that resist repeated efforts to answer them. Some can be understood by a good secondary-school student, while others are delicate or technical. Their perceived importance ranges from high to low, but they are alike in three ways. First, they did not arise from earlier mathematics, but in the course of practical investigations into diverse fields of science and engineering, some of them old and well-established, others newly revived, and a few of them altogether new. Second, they originate in careful inspection of actual pictures that were generated by computer. Third, they built upon the century-old mathematical "monster shapes" that were for a long time guaranteed to lack any contact with the real world.

The scope of this paper is necessarily limited. Many other fractal challenges and/or conjectures remain unanswered. Still others have been met and/or confirmed (especially in the context of multifractals). Among fields of research, fractal geometry seems to exemplify the shortest distance and the greatest contrast between a straightforward core, which is by now known to children and adult amateurs, and multiple frontiers filled with every kind of major difficulty, some of them linked with practical problems and other of purely mathematical interest.

1 Introduction

For three reasons listed in the abstract, the unanswered questions raised in this paper bear on an issue of great consequence. Does pure (or purified) mathematics exist as an autonomous discipline, one that can —and ideally should— adhere to a Platonic ideal and develop in total isolation from "sensations" and the "material" world? Or, to the contrary, is the existence of totally pure mathematics a myth?

In my work, the role of "sensations" is dominated by the role of fully-fledged pictures that are as detailed as possible and go well beyond sketches and diagrams. Their original goal was to help already formulated ideas and theories become accepted, by bridging cultural gaps between fields of science and mathematics. Then they went on to help me and many others generate new ideas and theories.

Many of these shapes strike everyone as being of exceptional and totally unexpected beauty. Some have the beauty of the mountains and clouds that they mean to represent; others seem wild and unexpected at first, but after brief inspection appear totally familiar. In front of our eyes, the visual geometric intuition built on the practice of Euclid and of calculus is being retrained with the help of new technology.

Pondering these pictures proves central to a different philosophical issue. What is beauty, and how does the beauty of these mathematical pictures relate to the beauty that a mathematician sees in his trade after long and strenuous practice? My lectures often underlined these questions, by showing what certain mathematical shapes really look like. By now, those pictures have become ubiquitous.

Next, consider the relation between pure mathematics and the "material" world. Everyone agrees that an awareness of physics, numerical experimentation and geometric intuition are very beneficial in some branches of mathematics, but elsewhere physics is reputed to be irrelevant, computation powerless, and intuition misleading. The irony is that history consistently proves that, as branches or branchlets of mathematics develop, they suddenly either lose or acquire deep but unforeseen connections with the sciences —old and new. As to numerical experimentation— which Gauss found invaluable but whose practice was waning until yesterday —it has seen its power multiply a thousandfold thanks to computers, and later, to computer graphics.

In no case that I know is this irony nearly as intense as in fractal geometry, a branch of learning that I conceived, developed and described in my book [FGN]. I put it to use in models and theories relative to diverse sciences, and it has become widely practiced. A "Polish school" of mathematics that had viewed itself as devoted exclusively to *Fundamenta*, added mightily to the list of the monster shapes, and greatly contributed to the creation of a chasm between mathematics and physics. Specifically ironical, therefore, is the fact that my work, that of my colleagues, and now that of many scholars, made those monster shapes, and new ones that are even more "pathological", into everyday tools of science.

This article uses freely the term *fractal*, which I coined in 1975 from the Latin word for "rough and broken up", namely *fractus*, and which is now generally accepted. Loosely, a "fractal set" is one whose detailed structure is a reduced-scale image of its overall shape. Among linear reductions, when the reduction ratio is the same in all directions, a fractal is "self-similar"; when those ratios differ, the fractal is self-affine. "Dust" will denote a totally disconnected set.

2 Complex Brownian bridge; Brownian cluster and its boundary; the self-avoiding plane Brownian motion

We begin with the open conjecture that is easiest to state and to understand.

Definitions. *The Wiener Brownian motion* $B(t)$ *is self-affine. Setting* $B(0)=0$, *recall that a* Brownian bridge $B_{bridge}(t)$ *is a periodic function of* t, *of period* 2π, *given for* $0 \leq t \leq 2\pi$ *by*

$$B_{bridge}(t) = B(t) - (t/2\pi)B(2\pi).$$

In distribution, $B_{\text{bridge}}(t)$ is identical to a sample of $B(t)$ conditioned to return to $B(0)=0$ for $t=2\pi$. It is the sum of Wiener's trigonometric series, whose n-th coefficient is G_n/n, where the G_n are independent reduced Gaussian random variables.

Take $B_{\text{bridge}}(t)$ to be complex of the form $B_r(t)+iB_i(t)$ and define a *Brownian plane cluster* Q as the set of values of $B_{\text{bridge}}(t)$. This non-traditional concept is the map of the time axis by the complex function $B_{\text{bridge}}(t)$. The classical map of the time axis by the complex $B(t)$ is everywhere dense in the plane, and the map of a time interval by the complex $B(t)$ is an inhomogeneous set. In contrast to the preceding example, when the origin Ω of the frame of reference belongs to Q, all the probability distributions concerning Q are independent of Ω; therefore Q is a *conditionally homogeneous* set.

The *self-avoiding planar Brownian motion* $\tilde{Q}$ is defined in [FGN] as being the closed set of points in Q accessible from infinity by a path that fails to intersect Q.

The unanswered "4/3 conjecture." The set $\tilde{Q}$ has a fractal dimension of $4/3$, in some suitable sense: Hausdorff-Besicovitch, or perhaps Bouligand ("Minkowski"), Tricot ("packing"), and/or other.

Comment. The original illustration of Q in Plate 243 of [FGN] is reproduced as Figure 1. It looked to me like an island with an especially wiggly coastline, hence visual intuition nourished by experience in the sciences suggested $D \sim 4/3$. This value was confirmed by direct numerical tests I commissioned and by further indirect numerical tests.

Literature: It is extensive and endowed with its own web site: math.duke.edu/faculty/lawler. Major contributions include C. Burdzy, G.F. Lawler, W. Werner, E.E. Puckette, and C. Bishop, P. Jones, R. Permantle & Y. Peres, (*J. Functional Analysis* in press).

Comments on the dimension 4/3, self-avoidance and squigs. There are two reasons for the term "self-avoiding Brownian motion": by definition, $\tilde{Q}$ does not self-intersect and its conjectured dimension $4/3$ is the value found in the self-avoiding random walk on a lattice. The latter $4/3$, which is unquestioned, was obtained by analytic arguments that are geometrically opaque, and the interpretation of $4/3$ as a dimension implies yet another unproven conjecture.

Squids and a wide open issue that combines fractals and topology. To avoid those difficulties, [FGN] (Chapter 24) introduced a class of recursive constructions, *squigs*, that create self-avoidance by recursive interpolation. The simplest is of dimension $\log 2.5/\log 2 \sim 1.3219...$; my original heuristic argument was confirmed by J. Peyrière (*C.R. Acad. Sc.* (Paris), **286**, 1978, 937 ; *Ann. Institut Fourier*: **31**, 1981, 187.) I suspect that the discrepancy between $4/3$ and 1.3219... follows from the fact that squigs involve a recursive subdivision or "triangulation" of the plane. Viewing this discrepancy as of secondary importance, I suspect that self-avoidance is linked in a profound and intrinsic way to the dimension $4/3$. The nature of this link is a mystery and a challenge.

3 Tools of fractal analysis: new, or old but obscure

Need to elaborate on the concept of fractal dimension. The preceeding section hinges on a single concept: a fractal dimension. This concept deserves several distinct comments. For the simplest self-similar fractals, D has a unique value, and to evaluate it is not much of a challenge, even for bright high-school students (if not younger). Those simplest examples contribute to the popularity of fractal geometry and to its pedagogical usefulness, but they happen to be utterly exceptional, hence very misleading. As to the Hausdorff measure, its definition is very short, and Hausdorff gave its value for the Cantor dusts. But it is known numerically in only a few other cases.

The multiplicity of dimensions of self-affine sets. After self-affine sets began to appear in concrete problems, several distinct fractal dimensions turned out to be needed. For example, the Hausdorff-Besicovitch (H.B.) dimension is a *local* concept, but *global* dimensions are also needed; they have been studied for very few cases, in references that are inaccessible but will be included in [SH]. Furthermore, the evaluation of the H-B dimension often proves extremely difficult (C. McMullen, Y. Peres, K. Falconer, to mention only a few) and conjectures abound. Even the graph of the Weierstrass function is of unknown H-B dimension.

The infinity of dimensions for a multifractal measure. While fractal sets call for a finite number of fractal dimensions, fractal measures call for an infinite number, in fact for a real function $f(\alpha)$ of a real variable α. This is one of the several reasons for calling such measures *multifractal.* Starting with the application to turbulence discussed in [SN], Multifractal measures occur in several areas of physics, and [SE] shows how they recently spread into finance. Besides, they are the topic of many studies of purely mathematical character. The literature is extremely large and there is no way of summarizing it here. But to assist the reader unacquainted with the topic and help introduce negative dimension, later in this section, it is good to briefly describe of the original random cascade multifractal.

Construction of a cascade multifractal. Given an integer base b, form the following array of independent and identically distributed (i.i.d.) random variables (r.v.): b r.v. $W(g)$, then b^2 r.v. $W(g,h)$, then b^3 r.v. $W(g,h,k)$ etc... Given a point $t \in [0,1]$, write it in base b as $t = 0, t_1t_2, \ldots t_n, \ldots$ Define $X'_n(t) = W(t_1)W(t_1,t_2)W(t_1,t_2,t_3)\ldots W(t_1,t_2,\ldots,t_n)$, and

$$X_n(t) = \int_0^t X'_n(s)ds.$$

In a paper reproduced in [SN] (in particular, *J. Fluid Mech.*, **62**, 1974, 331–358 and *C.R. Acad. Sc.* (Paris), **278A**, 1974, 289–292 & 355–358) I posed and solved in part many problems that are relative to a variety of classes of W's, and concern the weak or strong convergence of $X_n(t)$ to a non-vanishing limit $X(t)$, the numbers of finite moments of $X(t)$ and the dimension of the set of t's on which $X(t)$ varies. *Partial answers:* J.P. Kahane and J. Peyrière (*Adv. in Math.* **22**, 1976, 131–145 –translated in [SN]) confirmed and/or extended these conjectures and theorems. Here is one example: when $C = -EW \log_b W < 1$, the measure is non vanishing and can be said to be supported by a set of codimension C.

The fixed points of related smoothing transformations of probability distributions. Take b i.i.d.r.v. W_g, and b i.i.d.r.v. X_g having the same distribution as the $X(1)$ in the preceding paragraph. The weighted average $(1/b)\Sigma W_g X_{\hat{g}}$ (with the sum from 0 to $b-1$) has the same distribution as each X_g, meaning that $X(1)$ is a fixed point of the weighted averaging operation.

Negative dimensions and corresponding challenges and conjectures. Suppose now that the above-defined codimension C is > 1. If so, the measure almost surely vanishes and no further question about it was raised by mathematical analysis. Concrete needs, to the contrary, forced me to distinguished between several distinct levels of emptiness, and the dimension-like quantity $1 - C$, which is negative, is an excellent way of fulfilling this need. There is a discussion in *J. Fourier Analysis and Appl.* **Special issue**, 1995, 409–432.

Beyond all fractal dimensions. From the 1960s to the 1980s, the H-B dimension played an important role in helping fractal geometry be started and accepted. Today, the H-B dimension subsists as one of the many alternatives, at best a *primus inter pares.*

More important even is the fact that a careful analysis of both mathematical and observed fractals (in particular in the two sections that follow) showed the need for in many additional old or new tools. We now proceed to two examples.

Distinguishing between the Sierpinski curves on the basis of Urysohn-Menger ramification. Two ancient decorative designs occur in Sierpinski's investigations in the 1910's: one became known as the "carpet", and the second I called "gasket". Sierpinski used the carpet to show that a plane curve can be "topologically universal", that is, contain a homeomorphic transform of every other plane curve. The construction starts with a square, divides it into nine equal subsquares and erases the middle one, which I call a "trema" ($\tau\rho\eta\mu\alpha$ is the Greek term for "hole"). One proceeds in the same fashion with each remaining subsquare, and so on ad infinitum. As to the "gasket", Sierpinski used it to show that a curve can have branching points everywhere. The construction starts with an equilateral triangle, divides it into four equal subtriangles and erases the middle one as trema. One proceeds in the same fashion with each remaining subtriangle, and so on ad infinitum.

During the 1920's, the distinction between the carpet and the gasket became essential to the theory of curves. Piotr Urysohn and Karl Menger took them as prime examples of curves having, respectively, an infinite and a finite "order of ramification."

[FGN] quotes influential mathematicians who took the "gasket" as prime evidence that geometric intuition is powerless, because it can only conceive of branch points as being isolated, not everywhere dense. In fact, Gustave Eiffel himself wrote (as I interpret him) that he would have made his Tower even lighter, with no loss of strength, had the availability and cost of finer materials allowed him to increase the density of double points. From the Eiffel Tower to the Sierpinski gasket is an intellectual step that intuition can be trained to take.

The theory of curves that studies carpets, gaskets and the order of ramification became a stagnant corner of mathematics. Where can one find the latest facts about these notions? The surprising answer is that these notions came to be viewed as "unavoidable", once they were introduced in the statistical physics of condensed matter. Once ridden of the cobwebs of abstraction, they prove to be very practical and enlightening geometric tools to work with (e.g, Gefen, Mandelbrot & Aharony *Phys. Rev. Lett* **45**, 1980, 855–858). Physicists make them the object of scores of articles, and invent scores of generalizations that were not needed in 1915.

A new fractal tool: lacunarity. As is well-known, the most standard construction of a Cantor dust proceeds recursively as follow. The "initiator" is the interval $[0, 1]$. Its first stage ends with a generator made of N subintervals, each of length r. In the second stage, each generator interval is replaced by N intervals of length r^2, etc... The resulting limit set arose in the study of trigonometric series, but first attracted wider interest because of its topological and measure-theoretical properties. From those viewpoints, all Cantor dusts are equivalent. Much later, Hausdorff introduced his generalized dimension; this and every other definition of dimension yield the similarity dimension $D = \log N / \log 1/r$, an expression that has become widely known: the value of dimension splits the topological Cantor dusts into finer classes of equivalence parametrized by D.

Fractal geometry began by showing those classes of equivalence to be of great concrete significance. In due time, it went further, because the needs of science rather than mathematics required an even finer subdivision. To pose a problem, consider the Cantor–like constructions stacked in Figure 2. In the middle line, $N = 2$ and $r = 4^{-1}$; k steps below the middle line, $N = 2^k$, $r = 4^{-k}$ and the generator intervals are uniformely spaced ; k steps above the middle line, $N = 2^k$, $r = 4^{-k}$, but the generator intervals are crowded close to the endpoints of $[0, 1]$. The Cantor dusts in this stack share the common value $D = 1/2$, but look totally different. The Latin word for hole being *lacunar,* motion down (or up) the stack is said to correspond to decreasing (or increasing) *lacunarity.*

Challenge. As $k \to \infty$, the bottom line becomes "increasingly dense" on $[0, 1]$, and the top line "increasingly close to two dots". Provide a mathematical characterization of this "singular" passage to the limit.

Second challenge. [FGN], Chapters 33 to 35, describes and illustrates several constructions that allow a control of lacunarity. However, for the needs of both mathematics and science, the differences between the resulting constructs remained to be quantified. The existing studies of this quantification show that it is not easy and also not unique. Special complications occur when all the reduction ratios are identical, like in Figure 2. Of the alternative methods investigated in the literature, one is based on the prefactor of the relation $M(R) = FR^D$ that yields the mass $M(R)$ contained in a ball of radius R.

Another method is based on the prefactor in the Minkowski content. I studied it in *Fractal Geometry and Stockasties* (ed. C. Bandt et al) Birkhauser, 1995, pp. 12–38.

A third method has the advantage that defines a neutral level of lacunarity that separates positive and negative levels. On the line, this level is achieved by any randomized Cantor dust S with the following property. Granted that any choice of origin Ω in S divides the line into a right and a left half lines, lacunarity is said to be neutral when the intersections of S by those half lines are statistically independent. Increasingly positive (resp. negative) correlations are used to express and measure increasingly low (resp. high) levels of lacunarity. These notions will be used in the two sections that follow.

4 Major fractal clusters in statistical physics

While Brownian motion is fundamental in physics as well as in mathematics, the Brownian clusters in the first section are a mathematical curiosity. However, the property of fractality is shared by all the major real clusters (turbulence, galaxies, percolation, Ising, Potts) and all the major real interfaces (turbulent jets and wakes; metal and glass fractures; diffusion fronts). Each of these categories raises numerous open mathematical questions, of which a few will be listed.

Percolation clusters at criticality. (D. Stauffer & A. Aharony. *Introduction to Percolation Theory.* Second edition. London: Taylor & Francis.) Take an extremely large lattice of tiles. Each tile is chosen at random: with the probability p, it is made of vinyl and with the probability $1 - p$, of copper. Allow electric current to flow between two tiles if they have a side in common. A "cluster" is defined as a collection of copper tiles such that electricity can flow between two arbitrary points in the cluster. For an alternative, but equivalent, construction, define at the center of every tile, a random "relief function" $R(P)$ whose values are independent random variables uniformly distributed from 0 to 1. If this relief is flooded up to level p, each cluster stands out as a connected "island." Physicists conjectured, and mathematicians eventually proved, that there exists a "critical probability" p_C, such that a connected infinite island, i.e., a connected infinite conducting cluster, almost surely exist for $p < p_C$ but not for $p > p_C$.

The geometric complication of percolation clusters at criticality is extreme, and many of the basic conjectures arise not from pure thought, but careful examination of graphics.

Open conjecture A. Take an increasingly large lattice and resize it to be a square of unit side. At p_C, the infinite cluster converges weakly to a "limit cluster" that is a fractal curve.

Open conjecture B. The fractal dimension of this limit cluster is 91/48. This is the value obtained from a partly heuristic "field theoretical" argument that yields characteristic exponents.

Open conjecture C. The limit cluster is a finitely ramified curve in the sense of Urysohn-Menger.

Open conjecture D. Almost every linear cross-section of the limit cluster is a Lévy dust, as defined in [FGN]. Experimental evidence is found in Mandelbrot & Stauffer, *J. Physics* A 28, 1995, L 213 and Hovi, Aharony, Stauffer and Mandelbrot *Phys. Rev Lett.* **77**, 1996, 877–890.

The Ising model of magnets at the critical temperature. At each node of a regular lattice, the Ising model places a spin that can face up or down. The spins interact via forces between neighbors. By themselves, these forces create an equilibrium (minimum potential) situation in which all spins are either up or down. In addition, the system is in contact with a heat reservoir, and heat tends to invert the spins. When the temperature T exceeds a critical value T_C, heat overwhelms the interaction between neighbors. For $T < T_C$, local interactions between neighbors create global structures of greatest interest.

My work touched upon several issues in the shape of the up (or down) clusters at criticality.

Long open implicit question: Beginning with Onsager, it is known that in Euclidean space $\mathbb{R}^E$ the necessary and sufficient condition for magnets to exist is that $E > 1$. There are the innumerable mathematical differences between the $\mathbb{R}^E$ for $E = 1$ and $E > 1$. Identify differences that matter for the existence of magnets.

Partial answer: The specific examples of the Sierpinski curves and of related fractal lattices suggest that magnets can exist when and only when the order of ramification is infinite ([FGN], p. 139; Gefen, Mandelbrot and Aharony, *Phys. Rev. Lett* **45**, 1980, 855).

Conjecture: The above answer is of general validity.

Unanswered challenge. Rephrase the criterion of existence of magnets from the present indirect and highly computational form, to a direct form that would give a chance of proving or disproving the preceding conjecture.

Actual geometric implementation of the fractional-dimensional spaces of physics. Physicists are very successful with a procedure that is mathematically very dubious. They deal with spaces whose properties are obtained from those of Euclidean spaces by interpolation to "noninteger Euclidean dimensions." The dimension may be $4 - \varepsilon$ or $1 + \varepsilon$, where ε is in principle infinitesimal but is occasionally set to $\varepsilon = 1$. Calculations are carried out, in particular, expansions are performed in ε, and at the final stage, the "infinitesimal" ε is set to be the integer. Mathematically, these spaces remain unspecified, yet the procedure turns out to be extremely useful.

Mathematical challenge: Show that the properties postulated for those spaces are mutually compatible, show that they do (or do not) have a unique implementation; describe their implementation constructively.

Very partial solution: A very special example of such space has been implemented indirectly ([FGN], second printing, p. 462; Gefen, Meir, Mandelbrot & Aharony, *Phys. Rev. Lett.* **50**, 1983, 145). We showed that the postulated properties of certain physical problems in this space are identical to the *limits* of the properties of corresponding problems in a Sierpinski carpet whose "lacunarity" is made to converge to 0, in the sense that it tends to 0 as one moves down the stack on Figure 2.

5 The origin of fractality in partial differential equations

To establish that many features of nature (and, as shown in [SE], also of the Stock Market!) are fractal was the daunting task to which a large proportion of [FGN] is devoted. Important new examples keep being discovered, but the hardest present challenge is to discover the causes of fractality. Some cases remain obscure, but others are reasonably clear.

Thus, in the case of the physical clusters discussed in the preceeding section, fractality is the geometric counterpart of the techniques of statistical physics called scaling and renormalization, which show that the analytic properties of those objects follow a wealth of "power-law relations". Many mathematical issues, some of them already mentioned, remain open, but the overall renormalization framework is very firmly rooted.

Similarly, the study of dynamical systems features renormalization and resulting fractality in arguments that involve attractors, repellers and boundaries of basin of attraction. The fractal dimension of those boundaries directly affects the degree of sensitive dependence on initial conditions that characterizes chaotic dynamics. Renormalization also led to the Feigenbaum-Coullet-Tresser theory on bifurcations, and plays an important role in the study of complex quadratic maps (to be considered in a later section).

Unfortunately, additional examples of fractality proved to be beyond the usual renormalization. A notorious case concerns the diffusion-limited aggregates (DLA). Yet another source that covers many very important occurences of fractality led me to a very broad challenge-conjecture which was stated in [FGN], Chapter 11, and which will now be discussed.

Are smoothness and fractality doomed to coexist? *A quandary.* It is universally granted that physics is ruled by diverse partial differential equations, such as those of Laplace, Poisson, and Navier-Stokes. All differential equations imply a great degree of local smoothness, even though closer examination shows isolated singularities or "catastrophes". To the contrary, fractality implies everywhere dense roughness and/or fragmentation. This is one of the several reasons why fractal models in diverse fields were initially perceived as being "anomalies" that stand in direct contradiction with one of the firmest foundations of science.

A conjecture. There is no contradiction at all, in fact, fractals arise unavoidably in the long time behaviour of the solution of very familiar and "innocuous"–looking equations. In particular, many concrete situations where fractals are observed involve equations having free and moving boundaries, and/or interfaces, and/or singularities. As a suggestive "principle", [FGN] (Chapter 11) described the possibility that, under broad conditions that largely remain to be specified, those free boundaries, interfaces and singularities converge to suitable fractals.

Many equations were examined from this viewpoint, but two are of critical importance.

The large scale distribution of galaxies ; Newton's law and fractality. *Background.* Among astronomers, the near universally held view is that the distribution of galaxies is homogenous, except for local deviations. However ([FGN], Chapter 9), philosophers or science fiction writers played with the notion that the distribution is hierarchical, in a way unknowingly patterned along a spatial Cantor set. Hierarchical models were dismissed as unrealistic, in fact, largely forgotten. They are excessively regular, for example the reduction ratio must be (positive or negative) power of a basic ratio r_0. They necessarily imply that the Universe has a center and the model and reality can not only be matched by introducing a host of ad-hoc "patches". Last but not least, the hierarchical models predict nothing, that is, have no property that was not put in beforehand, and raise no new question.

Conjecture that the distribution of galaxies is properly fractal. ([FGN], Chapters 9 and 33 to 35). This conjecture results from a search for invariants that was central to every aspect of my construction of a fractal geometry. Granted that the distribution of galaxies certainly deviates in some ways from homogeneity, two broad approaches were tried. One consists in correcting for local inhomogeneity by incorporating local "patches". The next simplest global assumption is that the distribution is non-homogenous but scale-invariant. I chose to follow up this assumption, while excluding the strict hierarchies.

A surprising and noteworthy finding rewarded a detailed mathematical *and* visual investigation of sample sites generated by two concrete constructions of random fractal sets. As they are random, their self-similarity can only be statistical, which may be viewed as a drawback. But a more than counter-acting strong asset is that the self-similarity ratio can be chosen freely. It is not restricted to powers of a prescribed r_0, that is, the hierarchical structure is not a deliberate and largely arbitrary input. Quite to the contrary, the existences of clear-cut clusters are an unanticipated property of the construction. The details are given in [FGN]. The first construction is *The Seeded Universe,* based on a Lévy flight. Its Hausdorff-dimensional properties were known. Its correlation properties (Mandelbrot *C.R. Acad. Sc.* (Paris), **280A**, 1975, 1075) are nearly identical to those of actual galaxy maps. The second construction is *The Parted Universe*, which is obtained by subtracting from space a random collection of overlapping sets, already described as being called "tremas". Here, the tremas are allowed to overlap. Either construction yields sets that are highly irregular and involve no special center, yet exhibit a clear-cut clustering that was not deliberately inputted. They also exhibit "filaments" and "walls", which could not possibly have been inputted, because I did not know that they have been observed.

Conjecture that the observed "clusters", "filaments" and "walls" need not be explained separately, but necessarily follow from "scale free" fractality. This subtitle consists in conjecturing that the properties that it lists do not result from unidentified specific features of the models that have actually been studied, but follow as consequences from a variety of unconstrained forms of random fractality.

In the preceding title and the sentence that elaborates it, the word "conjecture" cannot be given its strict mathematical meaning, until a mathematical meaning is advanced for the remaining terms.

Lacunarity. A problem arose when careful simulations of the Seeded Universe proved to be visually far more "lacunar" than the real world. This notion, which was already mentioned, means that the simulations show the holes larger than in reality. The Parted Universe model fared better, since its lacunarity can be adjusted at will and fitted to the actual distribution, as shown in Mandelbrot, *C.R. Acad. Sc.* (Paris), **288**, 1979, 81–83.

A lowered lacunarity is expressed by a positive correlation between masses in antipodal directions. Testing this specific conjecture is a challenge for those who analyze the data.

Conjectured mathematical explanation of why one should expect the distribution of galaxies to be fractal. Consider a large array of point masses in a cubic box in which opposite sides are identified to form a 3 dimension' torus. How this array evolves under the action of inverse square attraction is a problem that obeys the Laplace equation, with the novelty that the singularities of the solution are the positions of the points, therefore, movable. All simulations I know of (beginning with those performed by IBM colleagues around 1960) suggest the following. Even when the pattern of the singularities begins by being uniform or Poisson, it gradually creates clusters and a semblance of hierarchy, and appears to tend toward fractality. It is against the preceeding background that I conjectured that the limit distribution of galaxies is fractal, and that the origin of fractality lies in Newton's equations.

The Navier Stokes and Euler equation of fluid motion and fractality of their singularities. *Background.* It is worth mentioning that the first concrete use of a Cantor dust in real spaces is found in a 1963 paper on noise records by Berger & Mandelbrot (reprinted in [SN]). This work was near simultaneous with Kolmogorov's work on the intermittence of turbulence. After numerous experimental tests, designed to create an intuitive feeling for this phenomenon (e.g., listening to turbulent velocity records that were made audible), I extended the fractal viewpoint to turbulence, and was led circa 1964 to the following conjecture.

Conjecture concerning facts. The property of being "turbulently dissipative" should *not* be viewed as attached to domains in a fluid with significant interior points, but as attached to fractal sets. In a first approximation, those sets' intersection with a straight line is a Cantor-like fractal dust having a dimension in the range from 0.5 to 0.6. The corresponding full sets in space should therefore be expected to be fractals with a Hausdorff dimension in the range from 2.5 to 2.6.

Actually, Cantor dust and Hausdorff dimension are not the proper notions in the context of viscous fluids, because viscosity necessarily erases the fine detail that is essential to Cantor fractals. Hence the following.

Conjecture: ([FGN], Chapter 11 and Mandelbrot, *C.R. Acad. Sc.* (Paris), **282A**, 1976, 119, translated as Chapter 19 of [SN]). The dissipation in a viscous fluid occurs in the neighborhood of singularity of a nonviscous approximation following Euler's equations, and the motion of a nonviscous fluid acquires singularities that are sets of dimension about 2.5 to 2.6. *Open mathematical problem:* To prove or disprove this conjecture, under suitable conditions.

Comment A. Several numerical tests agree with this conjecture (e.g. Chorin, *Comm. Pure and Appl. Math.*, **34**, 1981 853–866).

Comment B. I also conjectured that the Navier-Stokes equations have fractal singularities, of much smaller dimension. A technical inequality equivalent to this conjecture turned out to be present in a 1934 paper by J. Leray (*Acta Mathematica.*) Once revied and provided with the appropriate geometric interpretation, it led to extensive work by V. Scheffer, and then many others.

Comment C. As is well-known to students of chaos, a few years after my work, fractals in phase space entered in the study of the transition from laminar to turbulent flow, through the work of Ruelle & Takens and their followers. The task of unifying the real and phase-space roles of fractals is not yet completed.

6 Iterates of the complex map $z^2 + c$. Julia and Mandelbrot sets

The study of the iterates of rational functions of a complex variable reached a peak circa 1918. Fatou and Julia succeeded so well that – apart from the proof of the existence of Siegel discs – their theory remained largely unchanged for sixty years.

The J set or Julia set. This set, defined as the repeller of rational iteration, is typically a fractal: a nonanalytic curve or a "Cantor-like" dust. Julia called those repellers "very irregular and complicated." The computer –which I was the first to use systematically– reveals they are beautiful. To associate forever the name of Fatou and Julia, the *complement* of the Julia set is best called *Fatou set* and its maximal open components, *Fatou domains*. The wildly colorful displays that represent them must now be familiar to every reader.

Starting with the quadratic map $z \to z^2 + c$, I explored numerically how the value of c affects the nature of quadratic dynamics, and in particular, the shape of the Julia set.

The M_0 set. Of greatest interest from the viewpoint of dynamics, hence, of physics, is the set M_0 of those values of c for which $z^2 + c$ has a finite stable limit cycle.

The M_0 set having proved to be hard to investigate directly, I moved onto the computer-assisted investigation of a set that is easier to study, and seemed closely related.

The M set or Mandelbrot set. Douady & Hubbard gave this name to the set of those parameter values c in the complex plane, for which the Julia set is connected.

M, called μ-map in [FGN] (Chapter 19), proved to be a most worthy object of study, first for "experimental mathematics" and then for mathematics, and many facts are known about it. It even created a new form of art! It is so well and so widely known, that no further reference is needed. This discussion will limit itself to one major unsolved conjecture.

Conjecture that M is the closure of M_0. A computer approximation can only yield a set smaller than M_0, and a set larger than M. Extending the duration of the computation seemed to make the two representations converge to each other. Furthermore, when c is an interior point of M, not too close to the boundary, it was easily checked that a finite limit cycle exists. Those observations led to the conjecture that M is identical to M_0 together with its limit points.

In terms of its being simple and understandable without any special preparation, this conjecture comes close to where this paper starts: the "dimension $4/3$" conjecture about Brownian motion. Again, I could think of no proof, even of a heuristic one. More significantly, after eighteen-odd years, the conjecture remains unanswered.

The MLC conjecture. Many equivalent statements were identified, the best known being that the Mandelbrot set is locally connected. This statement acquired a "nickname", MLC; it has the great advantage of being local. (J.C. Yoccoz received high praise for proving it for a very large subset of the boundary). But, compared to the original form, it has the great drawback of being incomparably more sophisticated and, for most people, far from intuitive.

References

[FGN] Mandelbrot, B.B. 1982, *The Fractal Geometry of Nature*, W.H. Freeman and Co., New York and Oxford. The second and later printings include an Update and additional references. Earlier versions were *Les objets fractals: forme, hasard et dimension*, Flammarion, Paris, 1975 (fourth edition, 1995) and *Fractals: Form, Chance and Dimension*, Freeman, 1977.

[SE] Mandelbrot, B.B. 1997E, *Fractal and Scaling in Finance: Discontinuity, Concentration,Risk* (Selecta, Volume E) Springer-Verlag, New York.

[SN] Mandelbrot, B.B. 1998N, *Multifractals and 1/f Noise: Wild Self-Affinity in Physics.* (Selecta, Volume N). Springer-Verlag, New York (expected early in 1998).

[SH] Mandelbrot, B.B. 1998H, *Gaussian Fractals and Beyond* (Selecta, volume H). Springer-Verlag, New York (expected late in 1998).

Note: Specific references are added in the text.